

## EXACT STRING SOLUTIONS IN CURVED BACKGROUNDS

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We show how the classical string dynamics in  $D$ -dimensional curved background can be reduced to the dynamics of a massless particle constrained on a certain surface whenever there exists at least one Killing vector for the background metric. Then we obtain a number of sufficient conditions, which ensure the existence of exact solutions to the equations of motion and constraints. The results are also relevant to the null string case. Finally, we illustrate our considerations with an explicit example in four dimensions.

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## 1 Introduction

The string equations of motion and constraints in curved space-time are highly nonlinear and, *in general*, non exactly solvable [1]-[5]. Different methods have been applied to solve them approximately [6], [7], [8]-[10] or, if possible, exactly [11]-[15]. On the other hand, quite general exact solutions can be found by using an appropriate ansatz, which exploits the symmetries of the underlying curved space-time [13], [16]-[36]. In most cases, such an ansatz effectively decouples the dependence on the spatial world-sheet coordinate  $\sigma$  [16]-[29] or the dependence on the temporal world-sheet coordinate  $\tau$  [23, 26], [30]-[34]. Then the string equations of motion and constraints reduce to nonlinear coupled *ordinary* differential equations, which are considerably simpler to handle than the initial ones.

In this letter, we obtain some exact solutions of the classical equations of motion and constraints for both tensile and null strings in a  $D$ -dimensional curved background. This is done by using an ansatz, which reduces the initial dynamical system to the one depending on only one affine parameter. This is possible whenever there exists at least one Killing vector for the background metric. Then we search for sufficient conditions, which ensure the existence of exact solutions to the equations of motion and constraints without fixing particular metric. After that, we give an explicit example in four dimensional Kasner type background. Finally, we conclude with some comments on the derived results.

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## 2 Reduction of the dynamics

To begin with, we write down the bosonic string action in  $D$ - dimensional curved space-time  $\mathcal{M}_D$  with metric tensor  $g_{MN}(X)$

$$\begin{aligned} S &= \int d^2\xi \mathcal{L}, \quad \mathcal{L} = -\frac{T}{2} \sqrt{-\gamma} \gamma^{mn} \partial_m X^M \partial_n X^N g_{MN}(X), \\ \partial_m &= \partial/\partial \xi^m, \quad \xi^m = (\xi^0, \xi^1) = (\tau, \sigma), \quad m, n = 0, 1, \quad M, N = 0, 1, \dots, D-1, \end{aligned} \quad (1)$$

where, as usual,  $T$  is the string tension and  $\gamma$  is the determinant of the induced metric.

Here we would like to consider tensile and null (tensionless) strings on equal footing, so we have to rewrite the action (1) in a form in which the limit  $T \rightarrow 0$  can be taken. To this end, we set

$$\gamma^{mn} = \begin{pmatrix} -1 & \lambda^1 \\ \lambda^1 & -(\lambda^1)^2 + (2\lambda^0 T)^2 \end{pmatrix}$$

and obtain

$$\mathcal{L} = \frac{1}{4\lambda^0} g_{MN}(X) (\partial_0 - \lambda^1 \partial_1) X^M (\partial_0 - \lambda^1 \partial_1) X^N - \lambda^0 T^2 g_{MN}(X) \partial_1 X^M \partial_1 X^N.$$

The equations of motion and constraints following from this Lagrangian density are

$$\begin{aligned} \partial_0 \left[ \frac{1}{2\lambda^0} (\partial_0 - \lambda^1 \partial_1) X^K \right] - \partial_1 \left[ \frac{\lambda^1}{2\lambda^0} (\partial_0 - \lambda^1 \partial_1) X^K \right] \\ + \frac{1}{2\lambda^0} \Gamma_{MN}^K (\partial_0 - \lambda^1 \partial_1) X^M (\partial_0 - \lambda^1 \partial_1) X^N = \\ 2\lambda^0 T^2 \left[ \partial_1^2 X^K + \Gamma_{MN}^K \partial_1 X^M \partial_1 X^N + (\partial_1 \ln \lambda^0) \partial_1 X^K \right], \\ g_{MN}(X) (\partial_0 - \lambda^1 \partial_1) X^M (\partial_0 - \lambda^1 \partial_1) X^N = - (2\lambda^0 T)^2 g_{MN}(X) \partial_1 X^M \partial_1 X^N, \quad (2) \\ g_{MN}(X) (\partial_0 - \lambda^1 \partial_1) X^M \partial_1 X^N = 0, \quad (3) \end{aligned}$$

where

$$\Gamma_{MN}^K = \frac{1}{2} g^{KL} (\partial_M g_{NL} + \partial_N g_{ML} - \partial_L g_{MN})$$

is the connection compatible with the metric  $g_{MN}(X)$ . We will work in the gauge  $\lambda^m = \text{constants}$  in which the Euler-Lagrange equations take the form

$$\begin{aligned} (\partial_0 - \lambda^1 \partial_1) (\partial_0 - \lambda^1 \partial_1) X^K + \Gamma_{MN}^K (\partial_0 - \lambda^1 \partial_1) X^M (\partial_0 - \lambda^1 \partial_1) X^N \\ = (2\lambda^0 T)^2 (\partial_1^2 X^K + \Gamma_{MN}^K \partial_1 X^M \partial_1 X^N). \end{aligned} \quad (4)$$

Now we are going to show that by introducing an appropriate ansatz, one can reduce the classical string dynamics to the dynamics of a massless particle constrained on a certain surface whenever there exists at least one Killing vector for the background metric. Indeed, let  $M = (\mu, a)$ ,  $\{\mu\} \neq \{\emptyset\}$  and let us suppose that there exist a number

of independent Killing vectors  $\eta_\mu$ . Then in appropriate coordinates  $\eta_\mu = \frac{\partial}{\partial X^\mu}$  and the metric does not depend on  $X^\mu$ . In other words, from now on we will work with the metric

$$g_{MN} = g_{MN}(X^a).$$

On the other hand, we observe that

$$X^M(\tau, \sigma) = F_\pm^M[w_\pm(\tau, \sigma)], \quad w_\pm(\tau, \sigma) = (\lambda^1 \pm 2\lambda^0 T)\tau + \sigma$$

are solutions of the equations of motion (4) in arbitrary  $D$ - dimensional background  $g_{MN}(X^K)$ , depending on  $D$  arbitrary functions  $F_+^M$  or  $F_-^M$  (see also [11] for the possible background independent solutions and their properties). Taking all this into account, we propose the ansatz

$$\begin{aligned} X^\mu(\tau, \sigma) &= C_\pm^\mu w_\pm + y^\mu(\tau), \quad C_\pm^\mu = \text{constants}, \\ X^a(\tau, \sigma) &= y^a(\tau). \end{aligned} \quad (5)$$

Inserting (5) into constraints (2) and (3) one obtains (the dot is used for  $d/d\tau$ )

$$\begin{aligned} g_{MN}(y^a) \dot{y}^M \dot{y}^N \pm 2\lambda^0 T C_\pm^\mu [g_{\mu N}(y^a) \dot{y}^N \pm 2\lambda^0 T C_\pm^\nu g_{\mu\nu}(y^a)] &= 0, \\ C_\pm^\mu [g_{\mu N}(y^a) \dot{y}^N \pm 2\lambda^0 T C_\pm^\nu g_{\mu\nu}(y^a)] &= 0. \end{aligned}$$

Obviously, this system of two constraints is equivalent to the following one

$$\begin{aligned} g_{MN}(y^a) \dot{y}^M \dot{y}^N &= 0, \\ C_\pm^\mu [g_{\mu N}(y^a) \dot{y}^N \pm 2\lambda^0 T C_\pm^\nu g_{\mu\nu}(y^a)] &= 0. \end{aligned} \quad (6)$$

Using the ansatz (5) and constraint (7) one can reduce the initial Lagrangian to get

$$L^{red}(\tau) \propto \frac{1}{4\lambda^0} \left[ g_{MN}(y^a) \dot{y}^M \dot{y}^N - 2(2\lambda^0 T)^2 C_\pm^\mu C_\pm^\nu g_{\mu\nu}(y^a) \right].$$

It is easy to check that the constraint (6) can be rewritten as  $g^{MN} p_M p_N = 0$ , where  $p_M = \partial L^{red} / \partial \dot{y}^M$  is the momentum conjugated to  $y^M$ . All this means that we have obtained an effective dynamical system describing a massless point particle moving in a gravity background  $g_{MN}(y^a)$  and in a potential

$$U \propto T^2 C_\pm^\mu C_\pm^\nu g_{\mu\nu}(y^a)$$

on the constraint surface (7).

Analogous results can be received if one uses the ansatz

$$X^\mu(\tau, \sigma) = C_\pm^\mu w_\pm + z^\mu(\sigma), \quad X^a(\tau, \sigma) = z^a(\sigma). \quad (8)$$

Now putting (8) in (2) and (3) one gets (' is used for  $d/d\sigma$ )

$$\begin{aligned} &\left[ (2\lambda^0 T)^2 + (\lambda^1)^2 \right] g_{MN} z'^M z'^N \\ &+ 4\lambda^0 T \left[ (2\lambda^0 T \mp \lambda^1) C_\pm^\mu g_{\mu N} z'^N + 2\lambda^0 T C_\pm^\mu C_\pm^\nu g_{\mu\nu} \right] = 0, \\ &\lambda^1 g_{MN} z'^M z'^N + \left[ (\lambda^1 \mp 2\lambda^0 T) C_\pm^\mu g_{\mu N} z'^N \mp 2\lambda^0 T C_\pm^\mu C_\pm^\nu g_{\mu\nu} \right] = 0. \end{aligned}$$

These constraints are equivalent to the following ones

$$\begin{aligned} g_{MN}(z^a)z'^M z'^N &= 0, \\ C_\pm^\mu \left[ g_{\mu N}(z^a)z'^N + \frac{2\lambda^0 T}{2\lambda^0 T \mp \lambda^1} C_\pm^\nu g_{\mu\nu}(z^a) \right] &= 0. \end{aligned} \quad (9)$$

The corresponding reduced Lagrangian obtained with the help of (8) and (9) is

$$L^{red}(\sigma) \propto \frac{(\lambda^1)^2 - (2\lambda^0 T)^2}{4\lambda^0} \left[ g_{MN}(z^a)z'^M z'^N - 2 \left( \frac{2\lambda^0 T}{2\lambda^0 T \mp \lambda^1} \right)^2 C_\pm^\mu C_\pm^\nu g_{\mu\nu}(z^a) \right]$$

and a similar interpretation can be given as before.

In both cases - the ansatz (5) and the ansatz (8), the reduced Lagrangians do not depend on  $y^\mu$  and  $z^\mu$  respectively, and their conjugated generalized momenta are conserved.

Let us point out that the main difference between tensile and null strings, from the point of view of the reduced Lagrangians, is the absence of a potential term for the latter.

Because the consequences of (5) and (8) are similar, our further considerations will be based on the ansatz (5).

### 3 Exact solutions

To obtain the equations which we are going to consider, we use the ansatz (5) and rewrite (4) in the form

$$g_{KL}\ddot{y}^L + \Gamma_{K,MN}\dot{y}^M \dot{y}^N \pm 4\lambda^0 T C_\pm^\mu \Gamma_{K,\mu N} \dot{y}^N = 0. \quad (10)$$

At first, we set  $K = \mu$  in the above equality. It turns out that in this case the equations (10) reduce to

$$\frac{d}{d\tau} [g_{\mu\nu}\dot{y}^\nu + g_{\mu a}\dot{y}^a \pm 2\lambda^0 T C_\pm^\nu g_{\mu\nu}] = 0,$$

i.e we have obtained the following first integrals (constants of the motion)

$$g_{\mu\nu}\dot{y}^\nu + g_{\mu a}\dot{y}^a \pm 2\lambda^0 T C_\pm^\nu g_{\mu\nu} = A_\mu^\pm = \text{constants}. \quad (11)$$

They correspond to the conserved momenta  $p_\mu$ . From the constraint (7) it follows that the right hand side of (11) must satisfy the condition

$$C_\pm^\mu A_\mu^\pm = 0.$$

Using (11), the equations (10) for  $K = a$  and the constraint (6) can be rewritten as

$$2\frac{d}{d\tau} (h_{ab}\dot{y}^b) - (\partial_a h_{bc}) \dot{y}^b \dot{y}^c + \partial_a V = 4\partial_{[a} (g_{b]\mu} k^{\mu\nu} A_\nu^\pm) \dot{y}^b \quad (12)$$

and

$$h_{ab}\dot{y}^a\dot{y}^b + V = 0, \quad (13)$$

where

$$h_{ab} \equiv g_{ab} - g_{a\mu}k^{\mu\nu}g_{\nu b}, \quad V \equiv A_\mu^\pm A_\nu^\pm k^{\mu\nu} + (2\lambda^0 T)^2 C_\pm^\mu C_\pm^\nu g_{\mu\nu},$$

and  $k^{\mu\nu}$  is by definition the inverse of  $g_{\mu\nu}$ :  $k^{\mu\lambda}g_{\lambda\nu} = \delta_\nu^\mu$ . For example, when  $g_{MN}$  does not depend on the coordinate  $y^q$

$$h_{ab} = g_{ab} - \frac{g_{aq}g_{qb}}{g_{qq}},$$

when  $g_{MN}$  does not depend on two of the coordinates (say  $y^q$  and  $y^s$ )

$$h_{ab} = g_{ab} - \frac{g_{aq}g_{ss}g_{qb} - 2g_{aq}g_{qs}g_{sb} + g_{as}g_{qq}g_{sb}}{g_{qq}g_{ss} - g_{qs}^2},$$

and so on.

At this stage, we restrict the metric  $h_{ab}$  to be a diagonal one, i.e.

$$g_{ab} = g_{a\mu}k^{\mu\nu}g_{\nu b}, \quad \text{for} \quad a \neq b. \quad (14)$$

This allows us to transform further equations (12) and obtain (there is no summation over  $a$ )

$$\frac{d}{d\tau} (h_{aa}\dot{y}^a)^2 + \dot{y}^a \partial_a (h_{aa}V) + \dot{y}^a \sum_{b \neq a} \left[ \partial_a \left( \frac{h_{aa}}{h_{bb}} \right) (h_{bb}\dot{y}^b)^2 - 4\partial_{[a} A_{b]}^\pm h_{aa}\dot{y}^b \right] = 0, \quad (15)$$

where we have introduced the notation

$$A_a^\pm \equiv g_{a\mu}k^{\mu\nu}A_\nu^\pm. \quad (16)$$

In receiving (15), the constraint (13) is also used after taking into account the restriction (14).

To reduce the order of the differential equations (15) by one, we first split the index  $a$  in such a way that  $y^r$  is one of the coordinates  $y^a$ , and  $y^\alpha$  are the others. Then we impose the conditions

$$\partial_\alpha \left( \frac{h_{\alpha\alpha}}{h_{aa}} \right) = 0, \quad \partial_\alpha (h_{rr}\dot{y}^r)^2 = 0, \quad \partial_r (h_{\alpha\alpha}\dot{y}^\alpha)^2 = 0, \quad A_\alpha^\pm = \partial_\alpha f^\pm. \quad (17)$$

The result of integrations, compatible with (13) and (14), is the following

$$\begin{aligned} (h_{\alpha\alpha}\dot{y}^\alpha)^2 &= D_\alpha (y^a \neq y^\alpha) + h_{\alpha\alpha} \left[ 2(A_r^\pm - \partial_r f^\pm) \dot{y}^r - V \right] = E_\alpha (y^\beta), \\ (h_{rr}\dot{z}^r)^2 &= h_{rr} \left\{ \left( \sum_\alpha -1 \right) V - \sum_\alpha \frac{D_\alpha}{h_{\alpha\alpha}} \right\} + \left[ \sum_\alpha (A_r^\pm - \partial_r f^\pm) \right]^2 = E_r (y^r), \end{aligned}$$

where  $D_\alpha$ ,  $E_\alpha$ ,  $E_r$  are arbitrary functions of their arguments, and

$$\dot{z}^r \equiv \dot{y}^r + \frac{\sum_\alpha}{h_{rr}} \left( A_r^\pm - \partial_r f^\pm \right).$$

To find solutions of the above equations without choosing particular metric, we have to fix all coordinates  $y^a$  except one. If we denote it by  $y^A$ , then the *exact* solutions of the equations of motion and constraints for a string in the considered curved background are given by

$$X^\mu(X^A, \sigma) = X_0^\mu + C_\pm^\mu (\lambda^1 \tau + \sigma) - \int_{X_0^A}^{X^A} k_0^{\mu\nu} \left[ g_{\nu A}^0 \mp A_\nu^\pm \left( -\frac{h_{AA}^0}{V^0} \right)^{1/2} \right] du, \quad (18)$$

$$X^a = X_0^a = \text{constants} \quad \text{for} \quad a \neq A, \quad \tau(X^A) = \tau_0 \pm \int_{X_0^A}^{X^A} \left( -\frac{h_{AA}^0}{V^0} \right)^{1/2} du,$$

where  $X_0^\mu$ ,  $X_0^A$  and  $\tau_0$  are arbitrary constants. In these expressions

$$h_{AA}^0 = h_{AA}^0(X^A) = h_{AA}(X^A, X_0^{a \neq A})$$

and analogously for  $V^0$ ,  $k_0^{\mu\nu}$  and  $g_{\nu A}^0$ .

## 4 Explicit example

In this section we give an explicit example of exact solution for a string moving in four dimensional cosmological Kasner type background. Namely, the line element is ( $X^0 \equiv t$ )

$$ds^2 = g_{MN} dX^M dX^N = -(dt)^2 + \sum_{\mu=1}^3 t^{2q_\mu} (dX^\mu)^2, \quad (19)$$

$$\sum_{\mu=1}^3 q_\mu = 1, \quad \sum_{\mu=1}^3 q_\mu^2 = 1.$$

For definiteness, we choose  $q_\mu = (2/3, 2/3, -1/3)$ . The metric (19) depends on only one coordinate  $t$ , which we identify with  $X^a = y^a(\tau)$  according to our ansatz (5). Correspondingly, the last two terms in (15) vanish and there is no need to impose the conditions (17). Moreover, the metric (19) is a diagonal one, so we have  $h_{aa} = g_{aa} = -1$ . Taking this into account, we obtain the exact solution of the equations of motion and constraints (18) in the considered particular metric expressed as follows

$$X^\mu(t, \sigma) = X_0^\mu + C_\pm^\mu (\lambda^1 \tau + \sigma) \pm A_\mu^\pm I^\mu(t), \quad \tau(t) = \tau_0 \pm I^0(t),$$

$$I^M(t) \equiv \int_{t_0}^t du u^{-2q_M} V^{-1/2}, \quad q_M = (0, 2/3, 2/3, -1/3).$$

Although we have chosen relatively simple background metric, the expressions for  $I^M$  are too complicated. Because of that, we shall write down here only the formulas for

the two limiting cases  $T = 0$  and  $T \rightarrow \infty$  for  $t \geq 0$ . The former corresponds to considering null strings (high energy string limit).

When  $T = 0$ ,  $I^M$  reads

$$I^M = \frac{1}{2 \left[ (A_1^\pm)^2 + (A_2^\pm)^2 \right]^{1/2}} \left[ \frac{t^{2/3-2q_M}}{(q_M - 1/3) A} {}_2F_1 \left( 1/2, q_M - 1/3; q_M + 2/3; -\frac{1}{A^2 t^2} \right) + \frac{t_0^{5/3-2q_M}}{q_M - 5/6} {}_2F_1 \left( 1/2, 5/6 - q_M; 11/6 - q_M; -A^2 t_0^2 \right) + \frac{\Gamma(q_M - 1/3) \Gamma(5/6 - q_M)}{\sqrt{\pi} A^{5/3-2q_M}} \right],$$

where

$$A^2 \equiv \frac{(A_3^\pm)^2}{(A_1^\pm)^2 + (A_2^\pm)^2},$$

${}_2F_1(a, b; c; z)$  is the Gauss' hypergeometric function and  $\Gamma(z)$  is the Euler's  $\Gamma$ -function.

When  $T \rightarrow \infty$ ,  $I^M$  is given by the equalities

$$\begin{aligned} I^0 &= \pm \frac{1}{4\lambda^0 T C_\pm^3} \left[ \frac{6}{C} t^{1/3} {}_2F_1 \left( 1/2, -1/6; 5/6; -\frac{1}{C^2 t^2} \right) - \frac{3}{2} t_0^{4/3} {}_2F_1 \left( 1/2, 2/3; 5/3; -C^2 t_0^2 \right) + \frac{\Gamma(-1/6) \Gamma(2/3)}{\sqrt{\pi} C^{4/3}} \right], \\ I^{1,2} &= \pm \frac{1}{4\lambda^0 T C_\pm^3} \left[ \ln \left| \frac{(1 + C^2 t^2)^{1/2} - 1}{(1 + C^2 t^2)^{1/2} + 1} \right| - \ln \left| \frac{(1 + C^2 t_0^2)^{1/2} - 1}{(1 + C^2 t_0^2)^{1/2} + 1} \right| \right], \\ I^3 &= \pm \frac{1}{2\lambda^0 T C_\pm^3 C^2} \left[ (1 + C^2 t^2)^{1/2} - (1 + C^2 t_0^2)^{1/2} \right], \end{aligned}$$

where

$$C^2 \equiv \frac{(C_\pm^1)^2 + (C_\pm^2)^2}{(C_\pm^3)^2}.$$

## 5 Comments and conclusions

In this letter we performed some investigation on the classical string dynamics in  $D$ -dimensional curved background. In Section 2 we begin with rewriting the string action in a form in which the limit  $T \rightarrow 0$  could be taken to include also the null string case. Then we propose an ansatz, which reduces the initial dynamical system depending on two worldsheet parameters  $(\tau, \sigma)$  to the one depending only on  $\tau$ , whenever the background metric does not depend at least on one coordinate. An alternative ansatz is also given, which leads to a system depending only on  $\sigma$ . In Section 3, using the existence of an abelian isometry group  $G$  generated by the Killing vectors  $\partial/\partial X^\mu$ , the problem of solving the equations of motion and two constraints in  $D$ -dimensional curved space-time  $\mathcal{M}_D$  with metric  $g_{MN}$  is reduced to considering equations of motion and one

constraint in the coset  $\mathcal{M}_D/G$  with metric  $h_{ab}$ . As might be expected, an interaction with a gauge field appears in the Euler-Lagrange equations. In this connection, let us note that if we write down  $A_a^\pm$  introduced in (16) as

$$A_a^\pm = A_a^\nu A_\nu^\pm,$$

this gives the correspondence with the usual Kaluza-Klein type notation and

$$g_{MN}dy^M dy^N = h_{ab}dy^a dy^b + g_{\mu\nu} (dy^\mu + A_a^\mu dy^a) (dy^\nu + A_b^\nu dy^b).$$

In the remaining part of Section 3, we impose a number of conditions on the background metric, sufficient to obtain exact solutions of the equations of motion and constraints. These conditions are such that the metric is general enough to include in itself many interesting cases of curved backgrounds in different dimensions. In Section 4, we give an explicit example of exact string solution in four dimensional Kasner type background.

## References

- [1] A. Larsen, N. Sánchez, *Strings and Multi-Strings in Black Hole and Cosmological Spacetimes*, in "New developments in string gravity and physics at the Planck energy scale", World Scientific, Singapore, 1995, hep-th/9504007.
- [2] H. de Vega, A. Larsen, N. Sánchez, *Circular Strings and Multi-Strings in de Sitter and Anti de Sitter Spacetimes*, in "New developments in string gravity and physics at the Planck energy scale", World Scientific, Singapore, 1995, hep-th/9504050.
- [3] H. de Vega, N. Sánchez, *String Theory in Cosmological Spacetimes*, in "School of Astrophysics 1994", 227-271, hep-th/9504098.
- [4] H. de Vega, *Strings in Cosmological Spacetimes and their Backreaction*, in "Los Angeles 1995, Future perspectives in string theory", 148-159, hep-th/9505158.
- [5] H. de Vega, N. Sánchez, *Lectures on String Theory in Curved Spacetimes*, in "Erice 1995, String gravity and physics at the Planck energy scale", 11-63, hep-th/9512074.
- [6] H. de Vega, N. Sánchez, Phys. Lett., B 197 (1987) 320.
- [7] M. Gasperini, N. Sánchez, G. Veneziano, Int. J. Mod. Phys. A 6 (1991) 3853; Nucl. Phys. B 364 (1991) 365.
- [8] H. de Vega, A. Nicolaidis, Phys. Lett. B 295 (1992) 214;
- [9] H. de Vega, I. Giannakis, A. Nicolaidis, Mod. Phys. Lett A 10 (1995) 2479, hep-th/9412081.
- [10] C. Lousto, N. Sánchez, Phys. Rev. D 54 (1996) 6399, gr-qc/9605015.
- [11] O. Mattos, V. Rivelles, Phys. Rev. Lett. 70 (1993) 1583, hep-th/9210116.



- [12] F. Combes, H. J. de Vega, A. V. Mikhailov, N. Sánchez, Phys. Rev. D 50 (1994) 2754, hep-th/9310073.
- [13] H. de Vega, N. Sánchez, Phys. Rev. D 50 (1994) 7202, hep-th/9406029.
- [14] A. Frolov, S. Handy, A. Larsen, Nucl. Phys. B 468 (1996) 336, hep-th/9602033.
- [15] J. Maldacena, H. Ooguri, *Strings in  $AdS_3$  and the  $SL(2,R)$  WZW Model. Part 1: The spectrum*, hep-th/0001053.
- [16] H. de Vega, I. Egusquiza, Phys. Rev. D 49 (1994) 763, hep-th/9309016.
- [17] A. Larsen, Phys. Rev. D 50 (1994) 2623, hep-th/9311085.
- [18] H. de Vega, A. Larsen, N. Sánchez, Nucl. Phys. B 427 (1994) 643, hep-th/9312115.
- [19] A. Larsen, Phys. Rev. D 51 (1995) 4330, hep-th/9403193.
- [20] A. Larsen, N. Sánchez, Phys. Rev. D 50 (1994) 7493, hep-th/9405026.
- [21] H. de Vega, A. Larsen, N. Sánchez, Phys. Rev. D 51 (1995) 6917, hep-th/9410219.
- [22] A. Larsen, N. Sánchez, Int. J. Mod. Phys. A 11 (1996) 4005, hep-th/9501102.
- [23] A. Larsen, N. Sánchez, Phys. Rev. D 54 (1996) 2801, hep-th/9603049.
- [24] M. Dabrowski, A. Larsen, Phys. Rev. D 57 (1998) 5108, hep-th/9706020.
- [25] H. de Vega, A. Larsen, N. Sánchez, Phys. Rev. D 58 (1998) 026001, hep-th/9803038.
- [26] A. Larsen, N. Sánchez, Phys. Rev. D 58 (1998) 126002, hep-th/9805173.
- [27] A. Larsen, A. Nicolaidis, Phys. Rev. D 60 (1999) 024012, gr-qc/9812059.
- [28] A. Frolov, A. Larsen, Class. Quant. Grav. 16 (1999) 3717, gr-qc/9908039.
- [29] A. Larsen, N. Sánchez, *Quantum Coherent String States and  $SL(2,R)$  WZWN Model*, to appear in Phys. Rev. D, hep-th/0001180.
- [30] A. Larsen, N. Sánchez, Phys. Rev. D 51 (1995) 6929, hep-th/9501101.
- [31] S. Kar, Phys. Rev. D 52 (1995) 2036, gr-qc/9503004.
- [32] A. Frolov, A. Larsen, Nucl. Phys. B 449 (1995) 149, hep-th/9503060.
- [33] A. Frolov, S. Handy, A. Larsen, Phys. Rev. D 54 (1996) 2483, hep-th/9511069.
- [34] H. de Vega, I. Egusquiza, Phys. Rev. D 54 (1996) 7513, hep-th/9607056.
- [35] H. de Vega, J. Mittelbrunn, M. Medrano, N. Sánchez, Phys. Rev. D 52 (1995) 4609, hep-th/9502049.
- [36] H. de Vega, I. Egusquiza, Class. Quant. Grav. 13 (1996) 1041, hep-th/9505029.